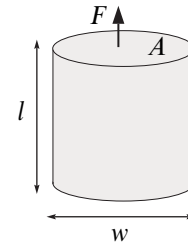


3.8 Elasticity

Elasticity definitions (simple)^a

Stress	$\tau = F/A$	(3.228)	τ stress F applied force A cross-sectional area
Strain	$e = \delta l/l$	(3.229)	e strain δl change in length l length
Young modulus (Hooke's law)	$E = \tau/e = \text{constant}$	(3.230)	E Young modulus
Poisson ratio ^b	$\sigma = -\frac{\delta w/w}{\delta l/l}$	(3.231)	σ Poisson ratio δw change in width w width



^aThese apply to a thin wire under longitudinal stress.

^bSolids obeying Hooke's law are restricted by thermodynamics to $-1 \leq \sigma \leq 1/2$, but none are known with $\sigma < 0$. Non-Hookean materials can show $\sigma > 1/2$.

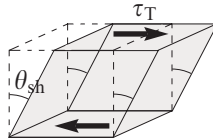
Elasticity definitions (general)

Stress tensor ^a	$\tau_{ij} = \frac{\text{force } \parallel i \text{ direction}}{\text{area } \perp j \text{ direction}}$	(3.232)	τ_{ij} stress tensor ($\tau_{ij} = \tau_{ji}$)
Strain tensor	$e_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$	(3.233)	e_{kl} strain tensor ($e_{kl} = e_{lk}$) u_k displacement $\parallel$ to x_k x_k coordinate system
Elastic modulus	$\tau_{ij} = \lambda_{ijkl} e_{kl}$	(3.234)	λ_{ijkl} elastic modulus
Elastic energy ^b	$U = \frac{1}{2} \lambda_{ijkl} e_{ij} e_{kl}$	(3.235)	U potential energy
Volume strain (dilatation)	$e_v = \frac{\delta V}{V} = e_{11} + e_{22} + e_{33}$	(3.236)	e_v volume strain δV change in volume V volume
Shear strain	$e_{kl} = \underbrace{\left(e_{kl} - \frac{1}{3} e_v \delta_{kl} \right)}_{\text{pure shear}} + \underbrace{\frac{1}{3} e_v \delta_{kl}}_{\text{dilatation}}$	(3.237)	δ_{kl} Kronecker delta
Hydrostatic compression	$\tau_{ij} = -p \delta_{ij}$	(3.238)	p hydrostatic pressure

^a τ_{ii} are normal stresses, $\tau_{ij} (i \neq j)$ are torsional stresses.

^bAs usual, products are implicitly summed over repeated indices.

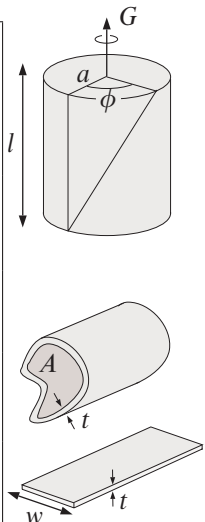
Isotropic elastic solids

Lamé coefficients	$\mu = \frac{E}{2(1+\sigma)}$	(3.239)	μ, λ Lamé coefficients
	$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$	(3.240)	E Young modulus σ Poisson ratio
Longitudinal modulus ^a	$M_l = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} = \lambda + 2\mu$	(3.241)	M_l longitudinal elastic modulus
Diagonalised equations ^b	$e_{ii} = \frac{1}{E} [\tau_{ii} - \sigma(\tau_{jj} + \tau_{kk})]$	(3.242)	e_{ii} strain in i direction τ_{ii} stress in i direction
	$\tau_{ii} = M_l \left[e_{ii} + \frac{\sigma}{1-\sigma} (e_{jj} + e_{kk}) \right]$	(3.243)	$\mathbf{e}$ strain tensor $\mathbf{t}$ stress tensor
	$\mathbf{t} = 2\mu\mathbf{e} + \lambda\mathbf{1}\text{tr}(\mathbf{e})$	(3.244)	$\mathbf{1}$ unit matrix $\text{tr}(\cdot)$ trace
Bulk modulus (compression modulus)	$K = \frac{E}{3(1-2\sigma)} = \lambda + \frac{2}{3}\mu$	(3.245)	K bulk modulus
	$\frac{1}{K_T} = -\frac{1}{V} \frac{\partial V}{\partial p} \Big _T$	(3.246)	K_T isothermal bulk modulus
	$-p = K e_v$	(3.247)	V volume p pressure T temperature
Shear modulus (rigidity modulus)	$\mu = \frac{E}{2(1+\sigma)}$	(3.248)	e_v volume strain
	$\tau_T = \mu\theta_{\text{sh}}$	(3.249)	μ shear modulus τ_T transverse stress θ_{sh} shear strain
Young modulus	$E = \frac{9\mu K}{\mu + 3K}$	(3.250)	
Poisson ratio	$\sigma = \frac{3K - 2\mu}{2(3K + \mu)}$	(3.251)	

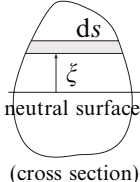
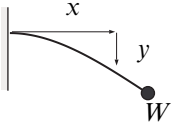
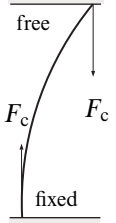
^aIn an extended medium.

^bAxes aligned along eigenvectors of the stress and strain tensors.

Torsion

Torsional rigidity (for a homogeneous rod)	$G = C \frac{\phi}{l}$	(3.252)	G twisting couple C torsional rigidity l rod length ϕ twist angle in length l	
Thin circular cylinder	$C = 2\pi a^3 \mu t$	(3.253)	a radius t wall thickness μ shear modulus	
Thick circular cylinder	$C = \frac{1}{2} \mu \pi (a_2^4 - a_1^4)$	(3.254)	a_1 inner radius a_2 outer radius	
Arbitrary thin-walled tube	$C = \frac{4A^2 \mu t}{P}$	(3.255)	A cross-sectional area P perimeter	
Long flat ribbon	$C = \frac{1}{3} \mu w t^3$	(3.256)	w cross-sectional width	

Bending beams^a

Bending moment	$G_b = \frac{E}{R_c} \int \xi^2 ds$	(3.257)	G_b bending moment	 (cross section)
	$= \frac{EI}{R_c}$	(3.258)	E Young modulus	
Light beam, horizontal at $x=0$, weight at $x=l$	$y = \frac{W}{2EI} \left(l - \frac{x}{3} \right) x^2$	(3.259)	R_c radius of curvature	
			ds area element	
Heavy beam	$EI \frac{d^4 y}{dx^4} = w(x)$	(3.260)	ξ distance to neutral surface from ds	
			I moment of area	
Euler strut failure	$F_c = \begin{cases} \pi^2 EI / l^2 & \text{(free ends)} \\ 4\pi^2 EI / l^2 & \text{(fixed ends)} \\ \pi^2 EI / (4l^2) & \text{(1 free end)} \end{cases}$	(3.261)	y displacement from horizontal	
			W end-weight	
			l beam length	
			x distance along beam	
			w beam weight per unit length	
			F_c critical compression force	
			l strut length	

^aThe radius of curvature is approximated by $1/R_c \simeq d^2 y / dx^2$.

Elastic wave velocities^a

In an infinite isotropic solid ^b	$v_t = (\mu / \rho)^{1/2}$	(3.262)	v_t speed of transverse wave
	$v_l = (M_1 / \rho)^{1/2}$	(3.263)	v_l speed of longitudinal wave
	$\frac{v_l}{v_t} = \left(\frac{2-2\sigma}{1-2\sigma} \right)^{1/2}$	(3.264)	μ shear modulus
In a fluid	$v_l = (K / \rho)^{1/2}$	(3.265)	ρ density
			M_1 longitudinal modulus ($= \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}$)
			K bulk modulus
On a thin plate (wave travelling along x , plate thin in z)	$v_l^{(x)} = \left[\frac{E}{\rho(1-\sigma^2)} \right]^{1/2}$	(3.266)	$v_l^{(i)}$ speed of longitudinal wave (displacement $\parallel i$)
	$v_t^{(y)} = (\mu / \rho)^{1/2}$	(3.267)	$v_t^{(i)}$ speed of transverse wave (displacement $\parallel i$)
	$v_t^{(z)} = k \left[\frac{Et^2}{12\rho(1-\sigma^2)} \right]^{1/2}$	(3.268)	E Young modulus
In a thin circular rod	$v_l = (E / \rho)^{1/2}$	(3.269)	σ Poisson ratio
	$v_\phi = (\mu / \rho)^{1/2}$	(3.270)	k wavenumber ($= 2\pi / \lambda$)
	$v_t = \frac{ka}{2} \left(\frac{E}{\rho} \right)^{1/2}$	(3.271)	t plate thickness (in z , $t \ll \lambda$)

^aWaves that produce “bending” are generally dispersive. Wave (phase) speeds are quoted throughout.

^bTransverse waves are also known as shear waves, or S-waves. Longitudinal waves are also known as pressure waves, or P-waves.

Waves in strings and springs^a

In a spring	$v_l = (\kappa l / \rho_l)^{1/2}$	(3.272)	v_l	speed of longitudinal wave
			κ	spring constant ^b
			l	spring length
			ρ_l	mass per unit length ^c
On a stretched string	$v_t = (T / \rho_l)^{1/2}$	(3.273)	v_t	speed of transverse wave
			T	tension
On a stretched sheet	$v_t = (\tau / \rho_A)^{1/2}$	(3.274)	τ	tension per unit width
			ρ_A	mass per unit area

^aWave amplitude assumed $\ll$ wavelength.

^bIn the sense $\kappa = \text{force}/\text{extension}$.

^cMeasured along the axis of the spring.

Propagation of elastic waves

Acoustic impedance	$Z = \frac{\text{force}}{\text{response velocity}} = \frac{F}{\dot{u}}$	(3.275)	Z	impedance
	$= (E' \rho)^{1/2}$	(3.276)	F	stress force
			u	strain displacement
Wave velocity/ impedance relation	if $v = \left(\frac{E'}{\rho} \right)^{1/2}$	(3.277)	E'	elastic modulus
	then $Z = (E' \rho)^{1/2} = \rho v$	(3.278)	ρ	density
			v	wave phase velocity
Mean energy density (nondispersive waves)	$\mathcal{U} = \frac{1}{2} E' k^2 u_0^2$	(3.279)	$\mathcal{U}$	energy density
	$= \frac{1}{2} \rho \omega^2 u_0^2$	(3.280)	k	wavenumber
	$P = \mathcal{U} v$	(3.281)	ω	angular frequency
			u_0	maximum displacement
			P	mean energy flux
Normal coefficients ^a	$r = \frac{u_r}{u_i} = -\frac{\tau_r}{\tau_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$	(3.282)	r	reflection coefficient
	$t = \frac{2Z_1}{Z_1 + Z_2}$	(3.283)	t	transmission coefficient
			τ	stress
Snell's law ^b	$\frac{\sin \theta_i}{v_i} = \frac{\sin \theta_r}{v_r} = \frac{\sin \theta_t}{v_t}$	(3.284)	θ_i	angle of incidence
			θ_r	angle of reflection
			θ_t	angle of refraction

^aFor stress and strain amplitudes. Because these reflection and transmission coefficients are usually defined in terms of displacement, u , rather than stress, there are differences between these coefficients and their equivalents defined in electromagnetism [see Equation (7.179) and page 154].

^bAngles defined from the normal to the interface. An incident plane pressure wave will generally excite both shear and pressure waves in reflection and transmission. Use the velocity appropriate for the wave type.